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Generic properties of a class of translation invariant quantum maps

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Abstract. The existence of delocalized regimes at all positive values of the Planck constant is proved to be a generic property of quantum maps of the 'kicked Harper' type. The qualitative properties of the delocalized wavepacket propagation are illustrated by numerical computations of the quantum-phase space distributions.

1. Introduction

Dynamical models described by maps are frequently used in several branches of quantum physics. Iterating a 'quantum map', i.e. a unitary propagator that defines a discrete-time dynamics, is numerically easier than solving a continuous-time Schrödinger equation; moreover, in spite of the crude approximations involved in reducing the exact dynamics to a map, the latter is often able to reproduce the essential qualitative features of the former; *in primis*, its stable or unstable nature.

The issue of stability is decided by the long-time behaviour of wavepackets in some representation, which is sometimes a coordinate representation but more often a momentum or an energy one. Wavepackets may oscillate quasi-periodically and thus remain localized forever inside some bounded region of the considered space; or else they can spread indefinitely. Perhaps the simplest case of such unbounded propagation occurs when the evolution operator is invariant under a group of translations in the relevant space; then the Bloch theorem applies and the spread of wavepackets in the relevant space is typically found to grow linearly in time. This is called ballistic propagation, and is generically associated with the existence of an absolutely continuous component in the spectrum of the evolution operator.

Translational invariance is linked with some strict periodicity. In momentum or energy space such a periodicity is often obtained under conditions of commensurability of some characteristic frequencies; in that case the ballistic propagation is also called quantum resonance. This is the case, e.g., with the kicked rotator model [1] when the period of the kicks is commensurate with the internal frequencies of the rotator.

Such outcomes for translational invariances are purely quantum effects and may be completely unrelated to classical dynamics, as demonstrated, for example, by the well known resonances occurring in the kicked rotator model [1].

The recently introduced 'kicked Harper' (KH) model provides grounds for the non-trivial application of the Bloch theorem [2]. It bears some resemblance to the conventional Harper model but unlike that model it has a classical dynamics marked by chaotic diffusion in appropriate parameter ranges. In its quantum version it has been numerically found to exhibit a variety of behaviours, including dynamical localization, ballistic propagation and diffusive-like propagation [2, 3]. The delocalized regimes are transparently related to translational invariance when the Planck constant is given certain 'commensurate' numerical values, but the occurrence of such regimes is by no means restricted to such special choices of $\hbar$.

In this paper we investigate a class of models of the KH type, parametrized by a 'potential' chosen in a class of analytic functions. We prove by a simple but nonetheless rigorous argument that the existence of delocalized regimes is a generic property in this class of models, independently of the arithmetic nature of $\hbar$. This we do by relating the spectral properties of the models to those of the same models taken 'on the line', where they are translationally invariant for any $\hbar$ (this translational invariance is not caused by any resonance of frequencies, anyway). We give the term 'generic' the same technical meaning as in the theory of (classical) differentiable dynamical systems, namely, we call 'generic' any property that is shared by all members of a set of the second Baire category in a suitable functional space. On mathematical grounds, very general methods such as the spectral duality theorem [4] have been developed for the class of almost periodic problems to which the KH model belongs. However, in the case of the KH model the crux of the matter lies with establishing the absence of infinitely degenerate quasi-energy eigenfunctions. Quite an analogous situation occurs in the theory of almost periodic Schrödinger operators [4], where infinite degeneracies have recently been ruled out [5]. In our case we prove that the occurrence of such infinite degeneracies is a non-generic property, but we actually suspect, on physical grounds, that such degeneracies should be absent in all cases.

In previous numerical works, the KH model was investigated on the circle, where it looks like a variant of the kicked rotator model. The KH model can also be analysed on the line by 'pasting' together the results of numerical simulations of a bundle of models on the circle, each one of which is taken at a different value of quasi-momentum. In the second part of this paper we report some numerical results obtained in this way. These results illustrate the evolution of the quantum-phase space distributions and suggest a simple classification of the different types of propagation.

2. Description of the model

We study a class of models with a discrete-time quantum dynamics generated by the iteration of a unitary propagator in $L^2(\mathbb{R})$ defined by

$$\hat{S} = e^{ikF(\hat{x})/\hbar} e^{-ilF(\hat{p})/\hbar} \tag{1}$$

where $F(x)$ is an analytic, real, even function with period 2π, and k, l are real non-negative parameters. (In the following it will sometimes be necessary to specify these data by writing $\hat{S} \equiv \hat{S}(F, k, l)$.)

The propagator (1) can be considered as a formal quantization of a classical map of the form:

$$\begin{aligned} x' &= x - lF'(p) \\ p' &= p + kF'(x') \end{aligned} \tag{1a}$$

which is an area-preserving map in the plane. Thanks to the periodicity of F, (1*a*) can also be used to define a map on the cylinder parametrized by $x(\mathrm{mod}(2\pi))$ and p or on the cylinder parametrized by x and $p(\mathrm{mod}(2\pi))$, and also on the torus parametrized by $x(\mathrm{mod}(2\pi))$, $p(\mathrm{mod}(2\pi))$. All these different dynamical systems are obtained from the map (1*a*) by 'folding back' its orbits onto cylinders or tori. This folding process has a quantum analogue; because the invariance of $\hat{S}$ under translations in x or in p by multiples of 2π allows for a Bloch decomposition that reduces $\hat{S}$ to a direct sum of operators, that are related to the classical cylindrical or toral maps.

Two cases have to be distinguished, according to whether $\hbar$ is incommensurate with 2π, or not. In the former case, the Bloch decomposition of $\hat{S}$ can be accomplished in either the p- or the x-representation. However, the corresponding operators of 'quasi-momentum' and 'quasi-position' do not commute with each other and the quasi-energies (i.e. the eigenphases of $\hat{S}$) can be expressed as functions of only one Bloch number.

Instead, if $\hbar$ is commensurate with 2π, certain subgroups of translations in x and in p commute with each other, and this yields quasi-energies that depend on two Bloch numbers.

We now need to introduce an appropriate formalism. For any wavefunction $\psi(x)$ in the x-representation, one can write

$$\psi(x) = \left(\frac{\hbar}{2\pi}\right)^{1/2} \int_0^1 \mathrm{d}\eta\, \mathrm{e}^{\mathrm{i}\eta x} \Psi_\eta(x) \tag{2}$$

where η is quasi-momentum and $\Psi_\eta(x)$ is a 2π-periodic function given by

$$\Psi_\eta(\theta) = \sum_{n=-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}n\theta} \hat{\psi}((n+\eta)\hbar) \tag{3}$$

$\hat{\psi}$ being the wavefunction in the p representation. Equation (2) defines a fibration of the Hilbert space in fibres labelled by the values of quasi-momentum. Since the latter is conserved under the dynamics (1), the operator $\hat{S}$ is itself fibred, according to

$$(\hat{S}\psi)(x) = \left(\frac{\hbar}{2\pi}\right)^{1/2} \int_0^1 \mathrm{d}\eta\, \mathrm{e}^{\mathrm{i}\eta x} (\hat{S}_\eta^{(p)} \Psi_\eta)(x). \tag{4}$$

The operator $\hat{S}_\eta^{(p)}$ will be called the p-fibre map at η. It acts in $L^2(0, 2\pi)$ according to

$$\hat{S}_\eta^{(p)} = \mathrm{e}^{\mathrm{i}kF(\hat{\theta})/\hbar} \mathrm{e}^{-\mathrm{i}lF((\hat{n}+\eta)\hbar)/\hbar} \tag{5}$$

where $\hat{n} = -\mathrm{i}\partial/\partial\theta$ with periodic boundary conditions.

A completely similar fibration is obtained by using the quasi-positions ξ in place of the quasi-momentum η. The x-fibre maps obtained in this way are related to the p-fibre maps via

$$\hat{S}_\alpha^{(p)}(F,k,l) = \hat{S}_\alpha^{(x)\dagger}(F,l,k). \tag{6}$$

The fibre maps are quite different objects from the original map (1). Anyone of them can be taken as a formal quantization of a classical cylindrical map; as such, it can be formally assumed to describe the dynamics of a particle on a circle, i.e. of a rotator.

The state of every such rotator is described by a wavefunction $\Psi_\eta(\theta)$ (or $\Psi_\xi(\theta)$). In the angular momentum representation, defined by the eigenfunctions $|n\rangle = e^{in\theta}$ the state of the rotator is specified by the non-normalized sequence $\hat{\psi}((n+\eta)\hbar)$ ($n \in \mathbb{Z}$) in the case of p-fibres and by the sequence $\psi((m+\xi)\hbar)$ in the case of x-fibres.

The evolution of any wavefunction $\psi(x)$ under the dynamics (1) can be obtained, by decomposing ψ in its fibres Ψ_η, and by letting every fibre evolve, independently of the others, under the action of the corresponding fibre map (5). The numerical results described in section 3 have been obtained in this way.

In the 'commensurate' case when $\hbar = 2\pi r/q$ (r, q integers) one further Bloch reduction is possible, that reduces (5) to a direct sum of unitary matrices of rank q, of the form:

$$\mathsf{M}_\xi \circ \mathcal{F} \circ \mathsf{N}_\eta \circ \mathcal{F}^{-1} \tag{7}$$

where $\mathcal{F}$ is the q-dimensional Fourier transform, and $\mathsf{M}_\xi, \mathsf{N}_\eta$ are diagonal matrices. The unitary matrices (7) provide a quantization of the classical toral map; such a quantization exists only for commensurate $\hbar$, and in that case there is a two-parameter family of non-equivalent quantizations labelled by ξ and η.

3. Mathematical results

The qualitative nature of the quantum dynamics described by $\hat{S}$ and that of the fibre dynamics described by the 'cylindrical' fibre maps $\hat{\mathcal{S}}$ is determined by their respective spectral properties. We shall investigate what spectral properties have to be expected for the operator $\hat{S}$ and for its fibre maps $\hat{\mathcal{S}}$, for a generic choice of the potential F in the class of analytic periodic functions. We shall say that a certain property of $\hat{S}(F)$ is generic in the class of analytic periodic Fs, if that property is shared by all Fs in a set of the second Baire category in a suitable space of analytic functions. Although the generic or non-generic character of a property depends on the choice of a class of functions, in the following we shall speak of generic properties of $\hat{S}(F)$ letting it be understood that the relevant functional class is that of analytic functions, properly defined in appendix 1.

In the case when $\hbar$ is commensurate with 2π the unitary matrix (7), i.e. the quantum 'toral' map will obviously have a discrete spectrum. For non-trivial choices of F, one at least of the eigenvalues will not be a constant on changing ξ, η† and it will therefore sweep a band in the spectrum of the p-fibres, x-fibres and the

† This can be readily checked by explicity writing the trace of the matrix (7).

complete operator as well. In the commensurate case therefore the operator $\hat{S}$ and the cylindrical operators $\hat{\mathcal{S}}$ will have a continuous component in their spectrum.

In the remainder of this section we shall be concerned with the incommensurate case: throughout the following discussion $\hbar/2\pi$ is assumed to be irrational. In that case there is still one Bloch number left for the map (1), that can be ξ or η depending on the chosen representation. On heuristic grounds one would therefore expect $\hat{S}$ to have a continuous spectrum. In this respect we have the following result.

Theorem 1. The following property of F is generic: $\hat{S}(F, 1, l)$ has a purely continuous spectrum for almost all $l \geqslant 0$, including $l = 1$.

'Almost all', in this statement and in the following, means except possibly for a set of zero Lebesgue measure. The proof of this theorem is given in appendix 1: we stress here that at the heart of this proof lies translational invariance, due to which any proper eigenvalue of $\hat{S}$ ought to be infinitely degenerate—a property that we prove to be non-generic.

The absence of infinite degeneracies is easily established for particular choices of F, that give rise to fibre maps which couple a finite number of momentum eigenstates. Such cases can be handled by a 'Wronskian' argument and provide a skeleton for the second category set constructed in appendix 1.

We can now prove a result that says how the continuity of the spectrum of $\hat{S}$ affects the spectra of its fibres $\hat{\mathcal{S}}$.

Lemma. Let $\hat{S}(F, k, l)$ have a purely continuous spectrum for given F, k, l. Then either $\hat{\mathcal{S}}_\eta^{(p)}(F, k, l)$ or $\hat{\mathcal{S}}_\xi^{(x)}(F, k, l)$ or both have a non-empty continuous spectrum for almost all η or ξ.

Proof. The spectra of $\hat{\mathcal{S}}_\alpha^{(p)}$ at different values of α are essentially the same. In fact, if η in equation (5) is taken as a random variable uniformly distributed in $[0, 2\pi]$, then the ergodic shift $\eta \to \eta + \hbar$ is mapped by the correspondence $\eta \to \hat{\mathcal{S}}_\eta^{(p)}$ in the shift $\hat{\mathcal{S}} \to \mathrm{e}^{-\mathrm{i}\theta}\hat{\mathcal{S}}\mathrm{e}^{\mathrm{i}\theta}$. Therefore, as a function of η the unitary operator $\hat{\mathcal{S}}_\eta^{(p)}$ is a random ergodic operator [6, 7] and it has been shown that under such conditions the continuous spectrum of $\hat{\mathcal{S}}_\eta^{(p)}$ is the same for almost all η [7].

Now suppose that $\hat{\mathcal{S}}_\eta^{(p)}(F, k, l)$ has a pure-point spectrum for almost all η; we have to prove that $\hat{\mathcal{S}}_\xi^{(x)}(F, k, l)$ has a non-empty continuous spectrum for almost all ξ. Consider the functions:

$$u_{nr}(x) = \chi_r(x)\mathrm{e}^{\mathrm{i}nx} \qquad r, n \in \mathbb{Z}$$

with χ_r the characteristic function of the interval $[r\hbar, (r+1)\hbar]$. Let $t \in \mathbb{Z}$ be discrete time and, for a given $\psi \in L^2(\mathbb{R})$, let $\psi(t) = \hat{S}^t\psi$. Then†

$$\mathcal{P}_r(t) \equiv \int_{r\hbar}^{(r+1)\hbar} |\psi(t, x)|^2 \mathrm{d}x = \sum_{n=-\infty}^{+\infty} |\langle u_{nr}|\psi(t)\rangle|^2 \equiv \sum_{n=-\infty}^{+\infty} |c_{nr}(t)|^2.$$

† We assume $\hbar < 2\pi$.

Let us introduce time averages up to time T:

$$\langle f(t)\rangle_T = \frac{1}{T}\sum_0^{T-1} f(t)$$

so that

$$\langle \mathcal{P}_r(t)\rangle_T = \sum_{n=-\infty}^{+\infty} \langle |c_{nr}(t)|^2\rangle_T. \tag{8}$$

Since the spectrum of $\hat{S}$ is continuous,

$$\lim_{T\to\infty} \langle |c_{nr}(t)|^2\rangle_T = 0 \qquad \forall n, r \tag{9}$$

and this entails that

$$\lim_{T\to\infty} \langle \mathcal{P}_r(t)\rangle_T = 0. \tag{10}$$

Interchanging the limit $T \to \infty$ and the infinite sum over n in equation (8) is the central point of this argument. Roughly speaking, this step is justified because the motion of the wavepacket has been assumed to be localized in p, so the distribution $|c_{nr}(t)|^2$ cannot spread arbitrarily in n. In appendix 2 we actually prove that (8) is uniformly convergent with respect to T.

Using the decomposition in x-fibres, we have

$$\lim_{T\to\infty} \langle \mathcal{P}_r(t)\rangle_T = \lim_{T\to\infty} \int_0^1 d\xi \langle |\langle r|\Psi_\xi^{(x)}(t)\rangle|^2\rangle_T = 0. \tag{11}$$

The limit for $T \to \infty$ of the quantity under the integral sign in equation (11) exists; therefore, (11) entails that this limit is 0 for almost all ξ. In other words, the dynamics of almost all x-fibres is such that the average probability of occupation of any state $|r\rangle$ decreases to zero in the limit $T \to \infty$. Being true for all r, this implies that almost all x-fibres have a non-empty continuous spectrum.

Together with theorem 1 and equation (6), the result just proven yields our main result:

Theorem 2. The following property of F is generic: for almost all $l \geqslant 0$ (including $l = 1$), either $\hat{\mathcal{S}}_\eta^{(p)}(F, 1, l)$ has a non-empty continuous spectrum for almost all η or $\hat{\mathcal{S}}_\eta^{(p)}(F, l, 1)$ has a non-empty continuous spectrum for almost all η. In particular $\hat{\mathcal{S}}_\eta^{(p)}(F, 1, 1)$ has a non-empty continuous spectrum for almost all η.

4. Some numerical results

The fibre dynamics of the KH model ($F(x) = \cos(x)$) has been numerically investigated in [2, 3]. Evidence was found that for $k > l$ the motion is delocalized with the expectation of n^2 growing in time as t^2; for $k < l$ localized motion has been usually observed, but in some cases the ballistic propagation found at $k > l$ was also observed on interchanging the values of k and l. At $k = l$ unbounded propagation was observed in all cases with $\langle n^2 \rangle \sim t^\alpha$ and α equal or close to 1; according to recent results, however, α may be increasing with k [8].

This behaviour is strongly suggestive of a singular continuous spectrum for which a multifractal analysis has been undertaken [2, 9]. All these results consistently fit into the generic picture provided by theorem 2.

A numerical investigation of the full dynamics described by the operator (1) with $F(x) = \cos(x)$ shows some elements of interest. In order to numerically compute the complete evolution (1) we have taken as initial state a coherent state centred near $x = 10, p = 10$. Having decomposed this state in p fibres, corresponding to $\sim 10^2$ evenly spaced quasi-momenta, we computed the corresponding fibre evolutions up to a time $t \sim 400$ and we used the results to reconstruct the complete wavefunction at time t. From this wavefunction we obtained the Husimi distribution $H(q,p,t) = |\langle z|\psi_t\rangle|^2$, with $|z\rangle$ the normalized coherent state at $z = q + ip$. The phase space pictures thus obtained are shown in figures 1–4. Figure 4 is about a commensurate case with $\hbar = \pi/2$. Here the phase distribution is spreading in time more or less in all directions.

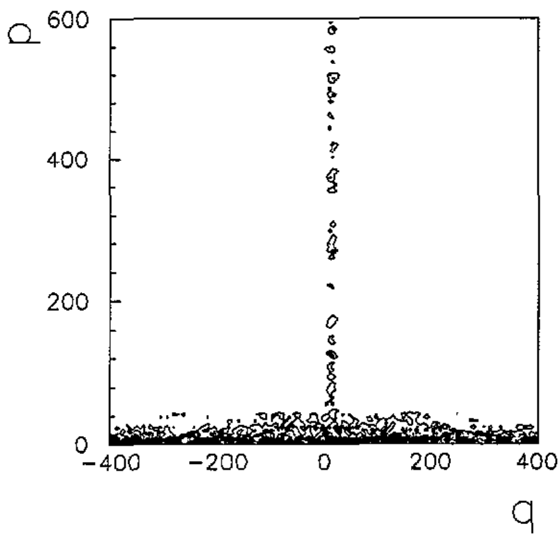

Figure 1. Contour plot of the Husimi distribution after 400 kicks for $k = 3.1$, $l = 6$, $\hbar = 2\pi/(6+\gamma)$, $\gamma = (\sqrt{5}+1)/2$ starting from a coherent state peaked in (10,10).

In all commensurate cases one can write

$$\psi(x,t) = \sum_{n=0}^{q-1} \int_0^1 d\xi \int_0^1 d\eta \, e^{i\lambda_n(\xi,\eta)t} \varphi_n(\xi,\eta,x)$$

with $\hbar = 2\pi p/q$. $\lambda_n(\xi,\eta)$ are the eigenphases of the Bloch matrix (7) and the smooth functions φ_n result from the expansion of $\psi(x,0)$ in Bloch waves. A stationary phase argument then shows that the Husimi distribution $H(q,p,t)$ decays in time as t^{-2}. Therefore the phase area (i.e. the semiclassical number of states) that

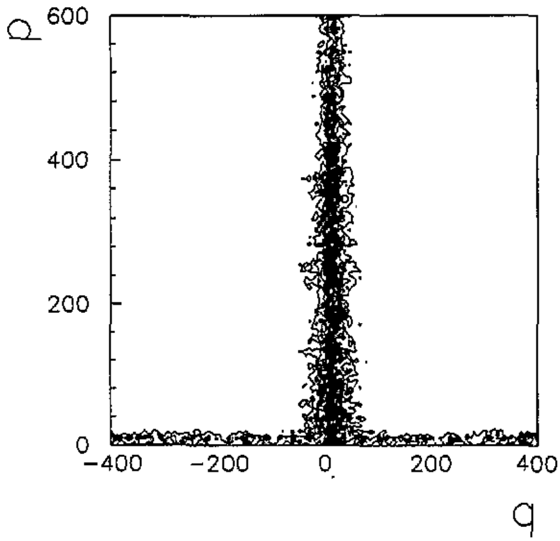

Figure 2. The same as figure 1, but with k and l interchanged.

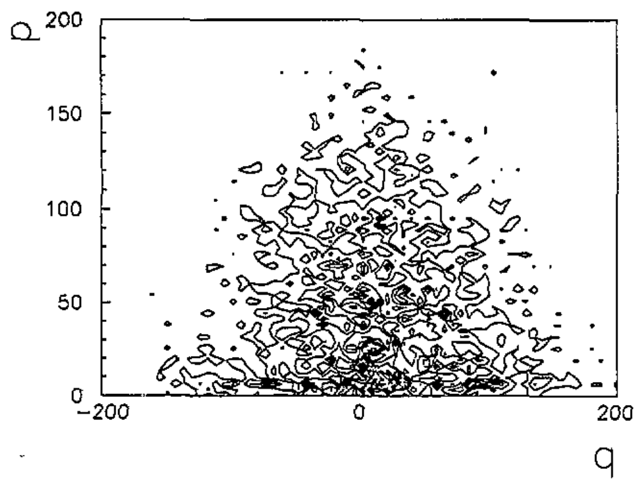

Figure 3. The same as figure 1, but with $k = l = 5$.

is significantly populated by H at time t can be assumed to grow as t^2, and since the propagation is (roughly) isotropic, the same type of growth is exhibited by $\langle x^2 \rangle$ and by $\langle p^2 \rangle$.

Figure 1 describes an incommensurate case with $k > l$. With the chosen values of k, l, delocalization occurs also with k and l interchanged (figure 2). In both cases the propagation is highly anisotropic and actually takes place along two 'chimneys' along the x- and p-axes. (The lack of symmetry between figures 1 and 2 which is observed on comparing the fine texture of the respective distributions is due to the fact that the discretization of the phase plane used to compute the Husimi distribution is not

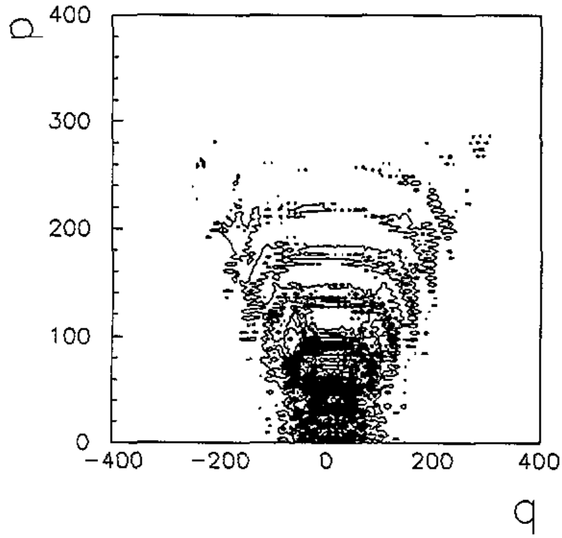

Figure 4. Husimi contour plot in the commensurate case $\hbar = \pi/2$, $k = l = 2$, after 100 kicks.

symmetric in x and p for technical reasons.)

Figure 3 describes an incommensurate case with $k = l$. In this case a rough isotropy is restored.

In the cases of figures 1 and 2 $\langle x^2 \rangle$ and $\langle p^2 \rangle$ increase as t^2; the same behaviour is exhibited by the fibre map. Instead, in figure 3 $\langle x^2 \rangle$ and $\langle p^2 \rangle$ increase over the explored time scale as t, which is also the behaviour of the fibre maps over the same time scale.

An interesting remark about these incommensurate cases is that the different types of propagation observed are consistent with a linear growth of the phase area in all cases, provided that account is taken of the different geometry of the propagation. Indeed, if the latter takes place inside 'chimneys' then a linear increase in the area enforces a quadratic growth of $\langle x^2 \rangle$ and $\langle p^2 \rangle$; if instead the distribution propagates in all directions, then $\langle x^2 \rangle$ and $\langle p^2 \rangle$ have to grow linearly, as observed. On the other hand, at the present time we do not know how the linear increase in the number of states could be theoretically justified. A formal stationary phase argument based on the existence of just one Bloch number would yield this result, but recourse to a stationary phase is hardly justified.

5. Concluding remarks

The results of section 2 admit some obvious generalizations. For instance, the proof of theorem 1 makes no use of the existence of two translation groups commuting with the map: one is sufficient to establish the result. By slight changes the argument can also be adapted to maps of the type $\exp i(F(\hat{x}) + tF(\hat{p}))/\hbar$ and, in particular, to the time-one propagator of the standard Harper model ($F(x) = \cos(x)$). In the latter case, the crucial step of ruling out infinite degeneracies is straightforward.

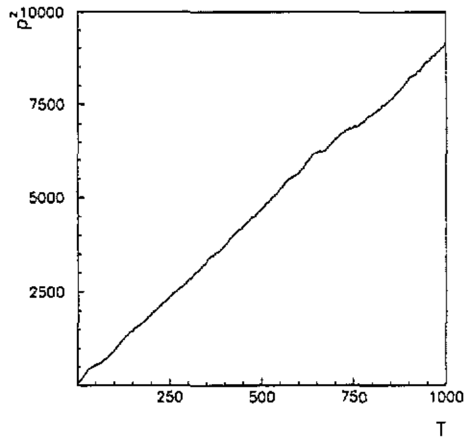

Figure 5. Expectation value of p^2 as a function of time for the complete map and for the same data as in figure 3.

Generally speaking, this very step plays a key role. Translation invariance alone does not preclude a point spectrum; for instance the Koopman operator associated with the classical map (1*a*), i.e. the unitary operator that describes the evolution of classical square-integrable phase functions, exhibits the same type of translation invariance as the quantum propagator, yet it can have an infinitely degenerate point spectrum due to the existence of KAM curves. The small overlap of quantum eigenfunctions provided by tunnelling is sufficient to remove any such degeneracy, generically at least; this physical argument actually suggests that our results may hold for all analytic potentials. However, to the best of our present knowledge this remains an open mathematical question.

Acknowledgments

We wish to thank F Izrailev and D Shepelyansky for discussions on the KH model, and J Bellissard for reading a draft of this paper.

Appendix 1

Let Γ_α the annulus in complex plane defined by $\alpha < |z| < \alpha^{-1}$ with $0 < \alpha < 1$. Let $\mathcal{B}_\alpha$ the vector space of real valued functions $F(\theta)$ defined on the unit circle that are even in $\theta = \arg(z)$, $(-\pi < \theta < \pi)$ and have an analytic continuation $\tilde{F}(z)$ in Γ_α, continuous in $\overline{\Gamma}_\alpha$. $\mathcal{B}_\alpha$ is a real Banach space under the norm

$$||F||_\alpha = \sup_{z\in\overline{\Gamma}_\alpha} |\tilde{F}(z)|.$$

We shall construct a subset $\mathcal{F} \subset \mathcal{B}_\alpha$, of the second Baire category in $\mathcal{B}_\alpha$, such that for all $F \in \mathcal{F}$ and for almost all $l > 0$ (including $l = 1$) the operator $\hat{S}(F, 1, l)$ has no proper eigenvalues. Any proper eigenvalue of $\hat{S}(F, 1, l)$ ought to be infinitely degenerate due to translational invariance. At the same time it ought to be an eigenvalue for almost all fibre maps; therefore, it will be sufficient to construct a second category set $\mathcal{F} \subset B_\alpha$ for no element of which the fibre maps have infinitely degenerate eigenvalues.

The determination of the eigenvectors of an operator of the type (5) can be formally reduced to the solution of a discrete Schrödinger equation. To this end we use a method introduced by Shepelyansky [9]. Let $\Phi(\theta)$ be an eigenfunction of the fibre map $\hat{\mathcal{S}}_\eta^{(p)}$:

$$\mathrm{e}^{\mathrm{i}F(\theta)/\hbar}\mathrm{e}^{-\mathrm{i}lF((\hat{n}+\eta)\hbar)/\hbar}\Phi(\theta) = \mathrm{e}^{-\mathrm{i}\omega}\Phi(\theta). \tag{A1.1}$$

Let $u(\theta) = \mathrm{e}^{\mathrm{i}F(\theta)/2\hbar}\Phi(\theta)g(\theta)$ with $g(\theta)$ a continuous real even function, and let u_n, $(n \in \mathbb{Z})$ the Fourier coefficients of $u(\theta)$. Then (A1.1) is equivalent to

$$\sum_{r=-\infty}^{+\infty} u_{n+r}W_r \sin(\chi_n + \Phi_r) = 0 \qquad \forall n \in \mathbb{Z} \tag{A1.2}$$

where

$$\chi_n = \omega - lF((n+\eta)\hbar)/\hbar$$

$$\mathrm{e}^{\mathrm{i}F(\theta)/2\hbar}g(\theta) = \sum_{r=-\infty}^{+\infty} W_r \mathrm{e}^{\mathrm{i}\phi_r}\mathrm{e}^{\mathrm{i}r\theta}$$

W_r are real and, on account of parity, $W_r = W_{-r}$. Equation (A1.2) is often likened to a solid-state one-dimensional model of the tight-binding type, with the coefficients W_r defining the 'hopping amplitude' between sites a distance r apart. We consider, first, those potentials F for which only finitely many W_r are different from zero. More precisely, let $\mathcal{I} \subset B_\alpha$ the class of potentials endowed with the following property:

Property I. A real even function $g(\theta)$ exists, such that g and $1/g$ are continuous, and $\mathrm{e}^{\mathrm{i}F(\theta)/2\hbar}g(\theta)$ is a polynomial in z and $1/z$ $(z = \mathrm{e}^{\mathrm{i}\theta})$.

For such potentials we can prove:

Proposition 1. If $F \in \mathcal{I}$, then $\forall l > 0$ no eigenvalue $\mathrm{e}^{-\mathrm{i}\omega}$ of (A1.1) can be infinitely degenerate.

Proof. If F has the property I then $W_r = 0$ for $|r| > N$, with a suitably chosen N. Let us define $2N$ vectors X_n by

$$X_n \equiv (u_{n+N-1}, u_{n+N-2}, \ldots, u_{n-N}) \tag{A1.3}$$

then equation (A1.2) yields

$$X_{n+1} = M^{(n)}X_n \tag{A1.4}$$

with the transfer matrix $M^{(n)}$ given by

$$M_{1,j}^{(n)} = \frac{W_{N-j}\sin(\chi_n + \Phi_{N-j})}{W_N \sin(\chi_n + \Phi_N)} \qquad (j = 1, 2, \ldots, 2N)$$

$$M_{i,j}^{(n)} = \delta_{i,j-1} \qquad (i > 1)$$

so that $\det(M^{(n)}) = 1$. Let $u^{(1)}, \ldots, u^{(2N)}$ be square-summable solutions of (A1.1) with the same $e^{-i\omega}$. The Fourier coefficients of everyone of them must satisfy the same equation (A1.2). Then let $X_n^{(j)}$ be the vectors (A1.3) defined by $u^{(j)}$. These vectors define matrices $S^{(n)}$ of rank $2N$ via $(S^{(n)})_{i,j} = (X_n^{(j)})_i$, and from (A1.4) we get

$$\det(S^{(n)}) = \det(S^{(n+1)}) \qquad \forall n$$

i.e. $\det(S^{(n)})$ does not depend on n. Now every $u_n^{(j)}$ is a square-summable sequence, so the vectors $X_n^{(j)}$ tend to 0 as $n \to \infty$; therefore, $\det(S^{(n)}) = 0$. Being true for all ns, this entails that $u^{(1)}, \ldots, u^{(2N)}$ are linearly dependent. (Note that $X_n^{(j)} = 0$ for some n would imply $X_n^{(j)} = 0$ for all n because of (A1.4).) □

Proposition 2. $\mathcal{I}$ is dense in $\mathcal{B}_\alpha$.

Proof. Given $F \in \mathcal{B}_\alpha$, let q an integer such that $||F||_\alpha < \pi q\hbar$. The function $w(z) = \tan(\tilde{F}(z)/2q\hbar)$ is analytic in $\overline{\Gamma}_\alpha$. By using Laurent expansion w can be arbitrary well approximated (in $\mathcal{B}_\alpha$) by polynomials $P(z)$ in z and $1/z$, real and even on the unit circle. Then $\tilde{F}(z) = 2q\hbar\tan^{-1}(w(z))$ can be likewise approximated by functions of the type $\tilde{F}_p(z) = 2q\hbar\tan^{-1}(P(z))$. Any such function has the property I; indeed,

$$e^{iF_p(\theta)/2\hbar} g(\theta) = \frac{(1 + iP(e^{i\theta}))^{2q}}{(1 + P^2(e^{i\theta}))^q} g(\theta)$$

and I will be satisfied by the choice $g(\theta) = (1 + P^2(e^{i\theta}))^q$ because P is real and even on the unit circle. □

From the dense set $\mathcal{I}$ we can extract a dense countable set $\mathcal{I}_0 = \{F_n\}$.
Given any vector $\psi \in L^2(R)$, let

$$R(F, T, l, \psi) = \frac{1}{T}\sum_{k=0}^{T-1} |\langle \hat{S}^k(F, 1, l)\psi|\psi\rangle|^2$$

and

$$\mathcal{R}(F, T, \psi) = \int_0^{+\infty} e^{-l'} R(F, T, l', \psi)\, dl'.$$

From proposition 1 it follows that for any $l \geqslant 0$ and for any n the operator $\hat{S}(F_n, 1, l)$ has a purely continuous spectrum. Therefore, as $T \to \infty$ both

$R(F_n, T, l, \psi)$ and $\mathcal{R}(F_n, T, \psi)$ tend to 0, $\forall n$. For all positive integer N and for any $\epsilon > 0$ let us define $\nu(N, \epsilon, \psi)$ as the smallest integer such that $\mathcal{R}(F_n, T, \psi) < \epsilon$ for $\forall T \geqslant \nu(N, \epsilon, \psi)$ and for $\forall n \leqslant N$; then $\nu(N, \epsilon, \psi)$ will monotonically diverge for $\epsilon \to 0$ and will be non-decreasing for $N \to \infty$.

Let σ_N, τ_N be arbitrary non-negative sequences tending to 0 for $N \to \infty$, and define

$$\gamma_N = \sigma_N / \nu^2(N, \tau_N) \tag{A1.5}$$

so that $\gamma_N \to 0$ for $N \to \infty$. Finally let us define subsets $\Lambda_{k,\psi}$ and $\mathcal{F}_\psi$ of $\mathcal{B}_\alpha$:

$$\Lambda_{k,\psi} \equiv \{F \in \mathcal{B}_\alpha : ||F - F_k|| < \gamma_k\}$$

$$\mathcal{F}_\psi = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \Lambda_{k,\psi}.$$

Proposition 3. $\mathcal{F}_\psi$ is a set of the second category in $\mathcal{B}_\alpha$ and $\forall F \in \mathcal{F}_\psi$, ψ belongs to the continuous subspace of $\hat{S}(F, 1, l)$ for almost all $l \geqslant 0$.

Proof. The set $\mathcal{F}_\psi$ is by definition a countable intersection of open sets, each one of which is dense (it contains all but finitely many of the F_ns). If $F \in \mathcal{F}_\psi$, a sequence $F_{n_k} \in \mathcal{I}_0$ can be found such that, $\forall k$,

$$||F - F_{n_k}||_\alpha < \gamma_{n_k}. \tag{A1.6}$$

The map $F \to \hat{S}(F, 1, l)$ is continuous from $\mathcal{B}_\alpha$ into the bounded operators in $L^2(\mathbb{R})$: indeed,

$$||\hat{S}(F, 1, l) - \hat{S}(F', 1, l)|| \leqslant c_1(1 + l)||F - F'||_\alpha$$

with a suitable constant c_1. Therefore, for any $T > 0$,

$$|\mathcal{R}(F, T, \psi) - \mathcal{R}(F', T, \psi)| \leqslant c_2 T ||F - F'||_\alpha.$$

From this and from equation (A1.6) we get, $\forall F \in \mathcal{F}_\psi$

$$\mathcal{R}(F, T, \psi) \leqslant c_2 T \gamma_{n_k} + \mathcal{R}(F_{n_k}, T, \psi). \tag{A1.7}$$

Now we can take $T_k = \nu(n_k, \tau_{n_k}, \psi)$ so that $T_k \to \infty$ by the definition of ν. From (A1.5) and (A1.7) we obtain

$$\mathcal{R}(F, T_k, \psi) \leqslant c_2 \sigma_{n_k} + \tau_{n_k} \to 0 \qquad \text{for } k \to \infty. \tag{A1.8}$$

On the other hand, the limit of $R(F, T, l, \psi)$ as $T \to \infty$ exists $\forall t$ and so does, by dominated convergence, the limit of $\mathcal{R}(F, T, \psi)$; the latter limit has to be zero on account of (A1.8). It follows that the limit of $R(F, T, l, \psi)$ is zero for almost all $l \geqslant 0$. Therefore ψ lies in the continuous subspace of $\hat{S}(F, 1, l)$ for almost all $l \geqslant 0$.

By a slight change in this argument we can define another set $\mathcal{F}'_\psi$ of the second category, such that $\forall F \in \mathcal{F}'_\psi$, ψ lies in the continuous subspace of $\hat{S}(F, 1, 1)$ (it suffices to define $\nu(N, \epsilon, \psi)$ from $R(F_n, T, 1, \psi)$.) Finally, the set

$$\mathcal{F} = \bigcap_n \left(\mathcal{F}_{\psi_n} \bigcap \mathcal{F}'_{\psi_n}\right)$$

where $\{\psi_n\}$ is a complete set of vectors, is a set of the second category, and for $F \in \mathcal{F}$ $\hat{S}(F, 1, l)$ has a purely continuous spectrum for a set of full measure of values of l, including $l = 1$.

Appendix 2

Upon decomposing $\chi_r(x)$ in its p-fibres $\tilde{\chi}_{r,\eta}(\theta)$ we get

$$c_{nr}(t) = \int_0^1 \mathrm{d}\eta \int_0^{2\pi} \mathrm{d}\theta\, \mathrm{e}^{-\mathrm{i}n\theta} \tilde{\chi}^*_{r\eta}(\theta) \Psi_\eta(\theta, t). \tag{A2.1}$$

Apart from a normalizing constant, the inner integral is the nth Fourier coefficient of $\tilde{\chi}^*_{r\eta}(\theta)\Psi_\eta(\theta,t)$. For any $\Phi \in L^2(0,2\pi)$ let us define

$$\gamma_n(\eta, \Phi) = \int_0^{2\pi} \mathrm{d}\theta\, \mathrm{e}^{-\mathrm{i}n\theta} \tilde{\chi}^*_{r\eta}(\theta)\Phi(\theta) \tag{A2.2}$$

(r is fixed throughout this argument). Then

$$\sum_{n=-\infty}^{+\infty} |\gamma_n(\eta, \Phi)|^2 = 2\pi \|\tilde{\chi}^*_{r\eta}\ \Phi\|^2 \tag{A2.3}$$

where the right-hand side is a continuous function of $\Phi \in L^2(0,2\pi)$ because $\tilde{\chi}^*_{r\eta}$ is a bounded function of θ. Therefore, from Dini's theorem we get that (A2.3) is uniformly convergent with respect to Φ varying on any relatively compact subset of $L^2(0,2\pi)$. Now the orbit $\{\Psi_\eta(t)\}_{t\in\mathbb{Z}}$ is just one such subset, (for almost all η), because the fibre dynamics is assumed to have a pure point spectrum, for almost all η. Therefore replacing Φ with $\Psi_\eta(t)$ in (A2.3) we conclude that the series thus obtained is uniformly convergent with respect to t for almost all η.

It follows that also the series

$$\sum_{n=-\infty}^{+\infty} \int_0^1 \mathrm{d}\eta |\gamma_n(\eta, \Psi_\eta(t))|^2 \tag{A2.4}$$

converges uniformly with respect to t (by monotone convergence). Therefore, the series $\sum_{n=-\infty}^{+\infty} |c_{nr}(t)|^2$, which is termwise dominated by (A2.4) is uniformly convergent too, and so is the series obtained by averaging up to time T.

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